Limits of the Turbine Efficiency for Free Fluid Flow

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An accurate estimate of the theoretical power limit of turbines in free fluid flows is important because of growing interest in the development of wind power and zero-head water power resources. The latter includes the huge kinetic energy of ocean currents, tidal streams, and rivers without dams. Knowledge of turbine efficiency limits helps to optimize design of hydro and wind power farms. An explicitly solvable new mathematical model for estimating the maximum efficiency of turbines in a free (nonducted) fluid is presented. This result can be used for hydropower turbines where construction of dams is impossible (in oceans) or undesirable (in rivers), as well as for wind power farms. The model deals with a finite two-dimensional, partially penetrable plate in an incompressible fluid. It is nearly ideal for two-dimensional propellers and less suitable for three-dimensional cross-flow Darrieus and helical turbines. The most interesting finding of our analysis is that the maximum efficiency of the plane propeller is about 30 percent for free fluids. This is in a sharp contrast to the 60 percent given by the Betz limit, commonly used now for decades. It is shown that the Betz overestimate results from neglecting the curvature of the fluid streams. We also show that the three-dimensional helical turbine is more efficient than the two-dimensional propeller, at least in water applications. Moreover, well-documented tests have shown that the helical turbine has an efficiency of 35 percent, making it preferable for use in free water currents. [DOI: 10.1115/1.1414137]

1 Modeling Turbines for Free Flow

1.1 The Betz Model for Rectilinear Flow. The efficiency limit of 59.3 percent was obtained by Betz back in the 1920s for propeller-type turbines in free flow. It became common practice to use this limit for estimating the maximum efficiency of such turbines, when designing wind farms. The derivation of the Betz limit can be found in many textbooks and other publications on fluid mechanics.

Betz considered a one-dimensional model for a plane turbine positioned in an incompressible fluid with rectilinear streams of constant velocity across any section of the current (Fig. 1(*a*)). The turbine was assumed to be under uniformly distributed pressure.

The efficiency of the turbine was defined as the ratio of the turbine power to the power of the unconstrained uniform flow through the turbine area. By basing his calculations on the momentum rate change and the Bernoulli relations for the fluid flowing through the turbine, Betz obtained an efficiency limit as high as 59.3 percent.

The principal assumption of the Betz model was that the fluid flow remains rectilinear when passing through the turbine and maintains a uniform distribution of the fluid pressure on the turbine. Such a distributed load leads to overestimating the forces and torque applied to the turbine and, as a result, to overestimating the turbine's power and its efficiency. In reality, the fluid streams are deflected from the rectilinear direction near the barrier, changing their motion to curvilinear trajectories and reducing their pressure on the turbine, as can be seen in Fig. 1(*b*): By taking account of the curvilinear trajectories for the streams, one obtains a more correct turbine power and efficiency limit.

1.2 Suggested Model for Curvilinear Flow. A new model (called the GGS model) for plane turbine in free flow with curvilinear streams is shown in Fig. 1(*b*). Comparison between the Betz and the GGS models demonstrates the following.

In the Betz model, Fig. 1(*a*), the resultant force is applied to each propeller at the center of pressure, which is a distance $R/2$ from the axis of the turbine, where R is the turbine radius. In the GGS model, Fig. 1(*b*), the resultant force is applied at the center of pressure that is calculated to be a distance $0.37R$ from the turbine axis, much closer to the turbine shaft. In both models, the lift and drag components of the resultant forces develop the torque that rotates the turbine. It is easily seen that the torque is greater in the Betz model than in the GGS model as a result of the difference in the lever arms.

Laboratory tests and measured efficiencies of operating turbines often confirm that the Betz limit is too high for both hydraulic and wind plane turbines. In particular, comparative performance of various hydraulic turbines in free flows shown in Fig. 2 supports the thesis that the Betz limit highly overestimates the propeller capacity when used in the water. The same comparison leads to the conclusion that the three-dimensional helical turbine would be preferable to any plane propeller in free water flows. The non-constrained helical turbine has exhibited an efficiency of 35 percent, for example, in well-documented hydraulic tests, and is superior to other known hydraulic turbomachines.

The mathematical formulation of the problem for plane turbine in free flow, its definitions, and exact solution will follow after Section 2.

2 Hydraulic Turbines for Free Flow

Practically all hydraulic turbines that are presently used for hydropower generation have been developed for installation in water dams across streams. This conventional design is the most economical and energy efficient for river hydropower plants because it provides maximum water heads and forces all the water to flow through the turbines under maximum hydraulic pressure. However, dams damage the environment and interfere with fish migration. They also cannot be used for power systems extracting energy from such huge potential sources as ocean currents or low-grade rivers. Thus, new hydraulic turbines are needed that can operate efficiently in free flow without dams.

For decades scientists and engineers have tried unsuccessfully to utilize conventional turbines for free and low-head hydro. The very efficient hydraulic turbines in high heads become so expensive in applications for low and ultralow-head hydroelectric stations that only very modest developments of this kind are found in

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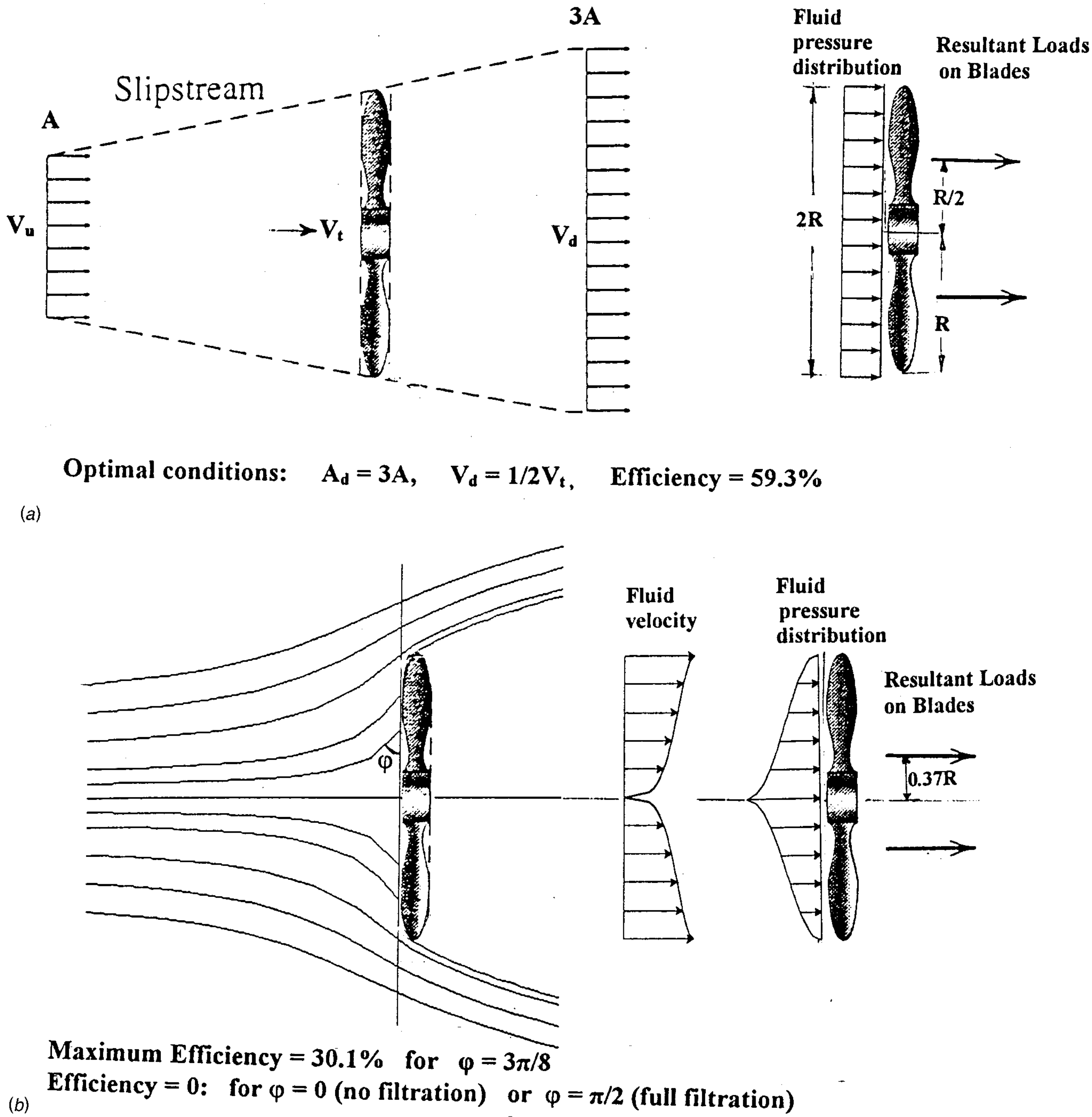
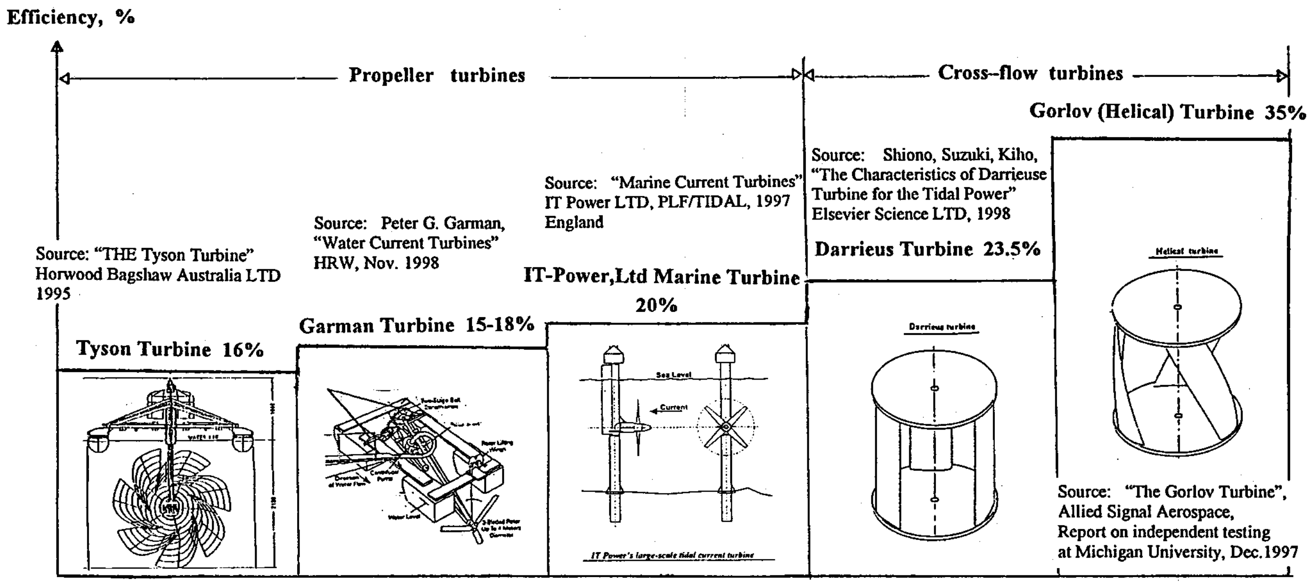


Fig. 1 Betz and GGS models for plane propeller in incompressible fluid flow—(*a*) Betz rectilinear flow model; (*b*) suggested curvilinear flow model ("GGS" model)

practice. For example, the unit cost of the Kaplan turbine jumps by a factor of 4 when the water head falls from 5–2 m.

The principal difference between exploiting high-head and free-flow turbines is that the latter need large flow openings to capture as much water masses as possible with low velocities and pressure. Conventional turbines, in contrast, are designed for high pressure and relatively small water ducts where all water has no chance to escape the turbine installed in the dam body. According to the Bernoulli theorem, the density of potential energy of flow is proportional to the pressure, while the density of the kinetic energy is proportional to the square of velocity. Conventional water turbines utilize mostly the potential component at the expense of the kinetic one. In order to do so, they need so-called "high solidity" where turbine blades cover most of the inside flow passage, resisting water flow and building up the water head. This causes the fluid velocity to fall and the kinetic component of Bernoulli equation to become negligibly small compared to the potential component. That is the reason why the higher water heads correspond to higher efficiency of hydraulic turbines, an efficiency that comes close to 90 percent in some cases. However, the situation is completely reversed for free water flows. In this case, the kinetic part dominates, and conventional turbines perform poorly, becoming very expensive.

Unlike the commonly used wheel-type turbines, the Darrieus reaction turbine for free flow, patented in 1931, has a barrelled shape with a number of straight or curved-in plane airfoil blades and a shaft that is perpendicular to the fluid flow. This turbine allows high torque to develop in slow flows, maintaining a large



Note: Some specific exploitation problems for Propeller and Darrieus turbines

1. **Propeller turbines with fixed blades cannot be used directly in reversible tidal flow as well as at shallow water sites**
2. **Darrieus turbines develop strong pulsation. They are not self- starting in most cases**

Fig. 2 Comparative performance of various turbines in free (nonducted) water currents

Fig. 3 Power systems for free flows with different helical turbines

water passage area. However, the Darrieus turbine has not received wide practical applications, mostly due to the pulsating during the rotation when blades change angles of attack traveling along the circular path. The turbine vibration often leads to the early fatigue failure of its parts and joints.

The new helical turbine, shown in Fig. 3, has all the advantages of the Darrieus turbine without its disadvantages, i.e., the helical turbine allows a large mass of slow water to flow through, captures its kinetic energy, and utilizes a very simple rotor, which is a major factor of its low cost [1,2]. The helical arrangement of the rotor blades eliminates the pulsation, improving its overall performance and leading to an efficiency as high as 35 percent that is substantially better than for other hydraulic machines in nonducted free flow, as shown in Fig. 2.

3 Definitions

Clearly, for a free flow turbine, the main problem is that any attempt to use the flow passing through the turbine more effectively would result in the increase of streamlining flow and might eventually decrease the net efficiency. The mathematical formulation of a free-flow turbine efficiency problem is discussed in this section, and an explicitly solvable model describing a certain class of flows is proposed in the next section.

The first important question of the efficiency problem can be formulated in terms of hydrodynamic resistance, disregarding the specific construction of the turbine. Denote the region where the turbines are located by Ω (assume that Ω is an open domain with a smooth or piecewise smooth boundary). Suppose also that the turbines are placed in a straight, uniform laminar current flowing towards the positive x-axis at velocity V_∞. The shape Ω is considered as a semi-penetrable obstacle for the stream with a resistance density r inside. That means that the filtration equation

$$-\nabla p = r\mathbf{V} \tag{1}$$

holds in Ω together with the continuity equation $\nabla \cdot \mathbf{V} = 0$, where p and $\mathbf{V}$ denote the pressure and the velocity of the flow, respectively. Denote the projection of Ω onto the yz-plane by Ω_n and its area by $|\Omega_n|$. The power carried by the flow through Ω_n is equal to

$$P_\infty = \frac{1}{2}\rho V_\infty^3 |\Omega_n| \tag{2}$$

In terms of density of hydrodynamic resistance, the power P consumed by the turbine is given by

$$P = \int_\Omega \nabla p \cdot \mathbf{V} = \int_\Omega \frac{1}{r}|\nabla p|^2 = \int_\Omega r|\mathbf{V}|^2 \tag{3}$$

by virtue of (1).

Definition. The *efficiency coefficient* $\mathcal{E}$ of a free-flow turbine is the ratio of the consumed power P to the power P_∞ carried by the flow through the projection of the turbine section region onto the plane perpendicular to it.

$$\mathcal{E} = \frac{P}{P_\infty} = \frac{\int_\Omega \nabla p \cdot \mathbf{V}}{\frac{1}{2}\rho V_\infty^3 |\Omega_n|} \tag{4}$$

The efficiency coefficient can be maximized by optimizing the resistance density. The optimal ratio between the streamlining current and the current passing through the turbines can be also obtained from this model. This parameter can be measured experimentally to determine how close a real turbine is to the theoretically optimal one.

If one were to use the model in the case of inviscid liquid, however, one would encounter the well-known d'Alambert paradox that an inviscid liquid meets no resistance from a streamlined obstacle. In the classical situation of streamlining, without the liquid penetrating through the obstacle, this paradox is resolved by considering a Helmholtz-type flow with separation [3,4]. This approach can be generalized for the case of a semi-penetrable obstacle, but the model, (1)–(4), should be slightly modified. The filtration equation (1) has to be localized on the part of the boundary of Ω near which the flow remains laminar, before it separates. Denote this part of the boundary by $\partial'\Omega$ and the surface density of resistance on it by r. In this case, the filtration equation takes the form

$$[p] = r\mathbf{V} \cdot \mathbf{n} \tag{5}$$

where $\mathbf{n}$ is the interior normal to $\partial'\Omega$ and $[p]$ is the pressure jump across $\partial'\Omega$. The power consumed by the turbine is given by

$$P = \int_{\partial'\Omega} [p]\mathbf{V} \cdot \mathbf{n} \tag{6}$$

and the expression for the efficiency becomes

$$\mathcal{E} = \frac{P}{P_\infty} = \frac{\int_{\partial'\Omega} [p]\mathbf{V} \cdot \mathbf{n}}{\frac{1}{2}\rho V_\infty^3 |\Omega_n|} \tag{7}$$

4 Modified Kirchhoff Flow in Application to the Problem of Free-Flow Turbine Efficiency

The classic Kirchhoff flow is a two-dimensional Helmoltz-type flow in which the current encounters a lamina placed perpendicularly to it [3]. Note that considering a two-dimensional model for the turbine efficiency could only increase the estimate, because the flow would become more constrained and might be closer to the actual situation for a shallow stream. On the other hand, a two-dimensional model allows us to apply conformal mapping methods, which cannot be used in higher dimensions, since any conformal map in $\mathbb{R}^n$ is the composition of a similarity transformation and an inversion if $n \geqslant 3$ [5].

The Argand diagram presenting the classic Kirchhoff flow is shown in Fig. 4. The stream separates from the edges forming a stagnation region past the lamina bounded by free streamlines γ and γ'. (These are symmetric to each other since the flow itself is symmetric.) Outside the stagnation region, the flow is potential. Let w be the complex potential of the flow, i.e., the complex analytic function defined in the flow domain (the complement to the stagnation domain), s.t. $\mathbf{V} = \overline{\partial w / \partial z}$. The condition $V = V_\infty$ is to be satisfied on free streamlines γ and γ'. This condition completes the setup of the free boundary problem, which can be solved by using the Kirchhoff transform [3,4] described in the forthcoming. The complex potential $w(z)$ maps the domain of the flow to the complement of the positive x-axis; see Fig. 4.

In the hodograph plane $\zeta = \xi + i\eta = \ln \partial w / \partial z$, the image of the flow domain is the semistrip $\{(\xi, \eta): -\pi/2 \leqslant \eta \leqslant \pi/2, -\infty \leqslant \xi \leqslant 0\}$ as shown in Fig. 4(c). In order to determine free streamlines γ and γ', the conformal map from ζ-plane to w-plane is constructed by means of Christoffel—Schwarz integral, which allows us to find the map $z(w)$. This method is also applicable in the case of partial penetration. For an arbitrary flow, its pictures in the w and ζ-planes might be rather complicated; but if one assumes that the flow crosses the lamina at the same angle at any point, the Kirchhoff transform is still convenient to use. This angle will be called the *pitch angle* and will be denoted by φ. The pictures of the flow in the z, w, and ζ-planes are shown in Fig. 5. Note that the units of length, time, and mass can be chosen in such a way that the density of the liquid, the breadth of the lamina, and the velocity of the flow at infinity are all equal to one.

5 Solution

In order to obtain the differential equation for the potential w, construct the conformal mapping ζ to w with the help of an auxiliary variable t. The map from the t-plane to the ζ-plane with the boundary extension as in Fig. 5 is given by

$$\zeta = -\left(1 - \frac{2\varphi}{\pi}\right)\ln\left(\frac{1}{t} + \frac{1}{t}\sqrt{1-t^2}\right) - i\left(\frac{\pi}{2} - \varphi\right) \tag{8}$$

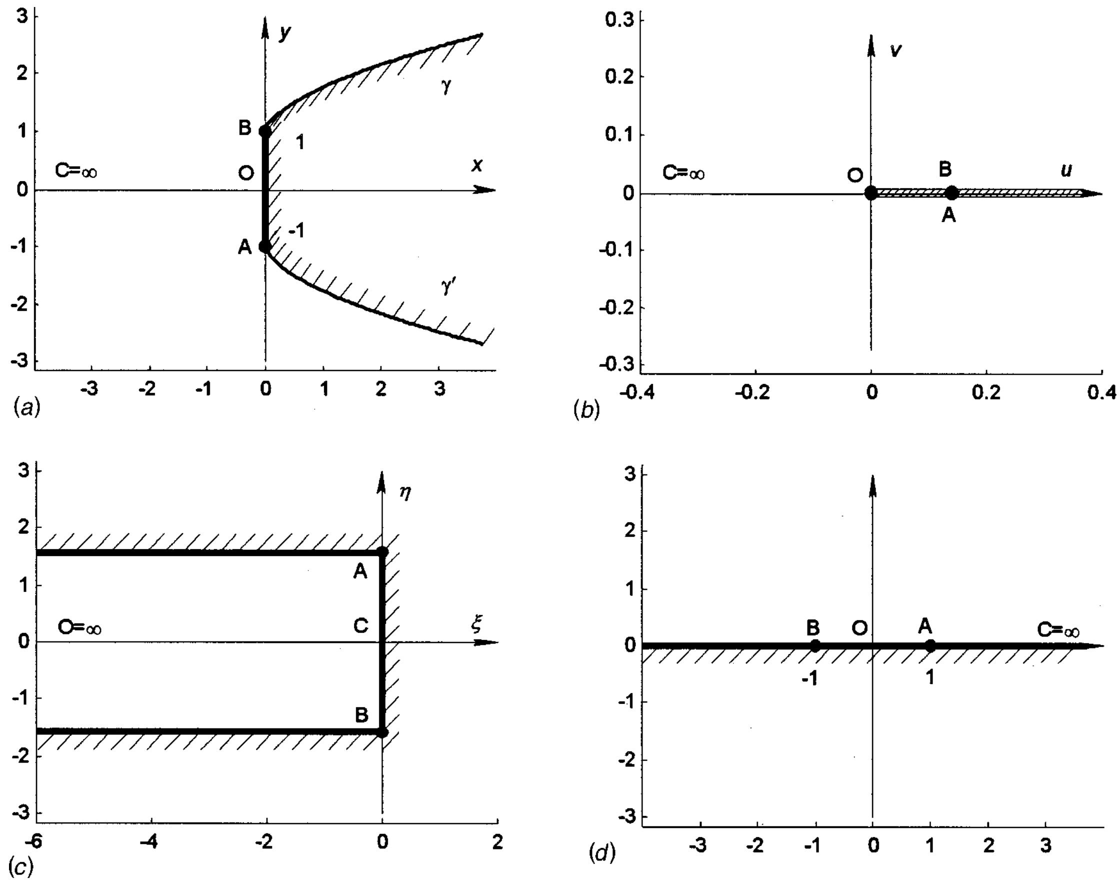


Fig. 4 Classic Kirchhoff flow—(*a*) *z*-plane, (*b*) potential *w*-plane, (*c*) hodograph *ζ*-plane, (*d*) *t*-plane

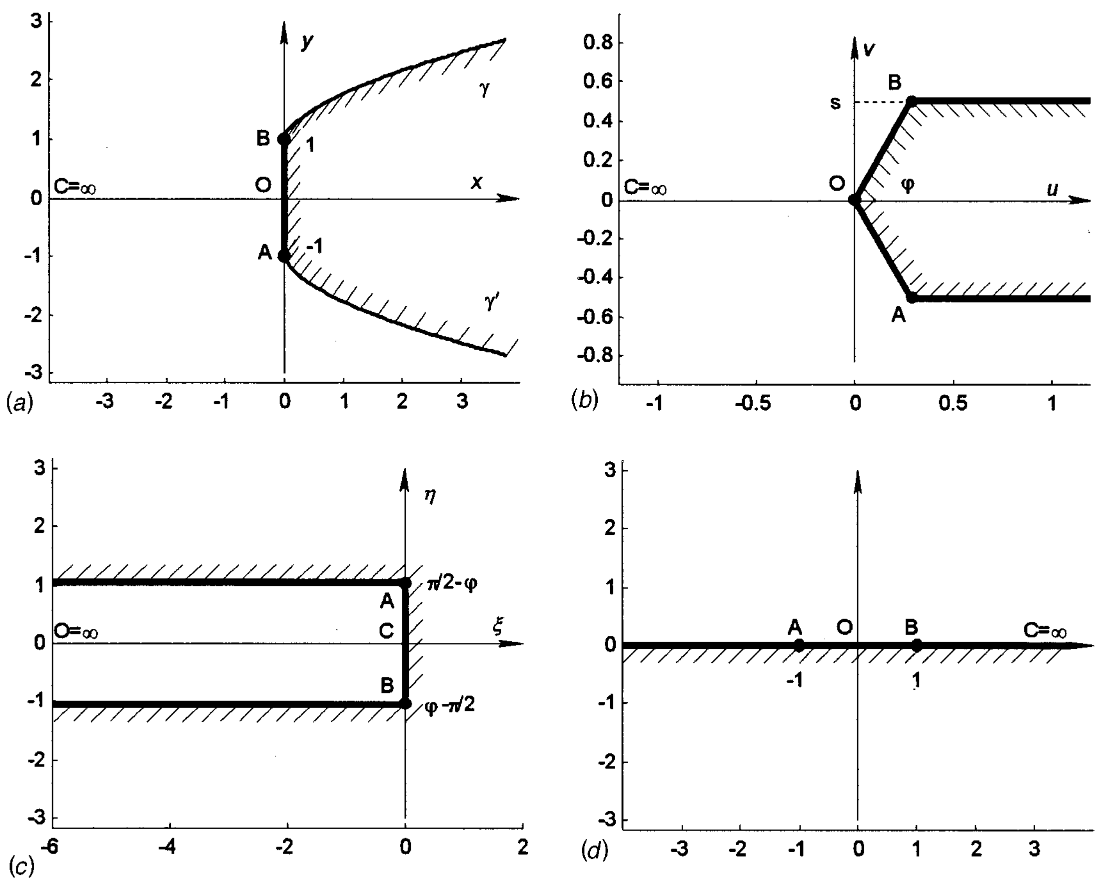


Fig. 5 Modified Kirchhoff flow—(*a*) *z*-plane, (*b*) potential *w*-plane, (*c*) hodograph *ζ*-plane, (*d*) *t*-plane

Table 1

No.	Pitch angle, φ	Efficiency, $\mathcal{E}$	Flow through, s
0	0.00000	0.00000	0.00000
1	0.07854	0.01761	0.02294
2	0.15708	0.03646	0.04785
3	0.23562	0.06922	0.09168
4	0.31416	0.07771	0.10405
5	0.39270	0.09998	0.13559
6	0.47124	0.12320	0.16961
7	0.54978	0.14717	0.20623
8	0.62832	0.17164	0.24562
9	0.70686	0.19625	0.28793
10	0.78540	0.22050	0.33333
11	0.86394	0.24371	0.38199
12	0.94248	0.26494	0.43409
13	1.02102	0.28292	0.48983
14	1.09956	0.29582	0.54940
15	1.17810	0.30113	0.61302
16	1.25664	0.29521	0.68091
17	1.33518	0.27274	0.75331
18	1.41372	0.22569	0.83044
19	1.49226	0.14158	0.91259
20	1.57080	0.00000	1.00000

The map from the t-plane to w-plane is constructed by means of a Christoffel—Schwarz integral

$$\frac{dw}{dt}=\frac{2s}{\varphi}(t^2-1)^{\varphi/\pi}t^{(1-2\varphi/\pi)} \tag{9}$$

$$w(t)=\frac{2s}{\varphi}\int_0^t(\tau^2-1)^{\varphi/\pi}\tau^{(1-2\varphi/\pi)}d\tau \tag{10}$$

Here, s is the width of the stagnation domain in the w-plane, which can also be interpreted as the distance between the free streamlines at infinity, or the fraction of the flow passing through the turbines. Since $\zeta=\ln dw/dz$, then (8) yields

$$\ln\frac{dz}{dw}=\left(1-\frac{2\varphi}{\pi}\right)\ln\left(\frac{1}{t}+\frac{1}{t}\sqrt{1-t^2}\right)+i\left(\frac{\pi}{2}-\varphi\right) \tag{11}$$

$$\frac{dz}{dw}=e^{i(\pi/2-\varphi)}(1+\sqrt{1-t^2})^{1-2\varphi/\pi}t^{2\varphi/\pi-1} \tag{12}$$

and

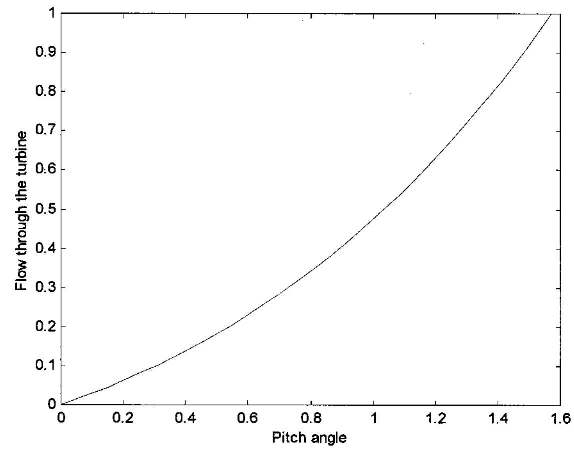


Fig. 7 Flow through s versus pitch angle φ

$$\frac{dz}{dt}=\frac{dz}{dw}\frac{dw}{dt}=\frac{2is}{\varphi}(1+\sqrt{1-t^2})^{1-2\varphi/\pi}(1-t^2)^{\varphi/\pi} \tag{13}$$

Using (13) and the fact that $\int_0^1 dz/dt\,dt=i$, one can compute s as

$$s=\frac{\varphi}{2I_2(\varphi)} \tag{14}$$

where

$$I_2(\varphi)=\int_0^1(1+\sqrt{1-t^2})^{1-2\varphi/\pi}(1-t^2)^{\varphi/\pi}dt \tag{15}$$

By virtue of the Bernoulli theorem, the pressure jump over the lamina is equal to

$$[p]=\frac{1}{2}(V_\infty^2-V^2) \tag{16}$$

and the expression for the efficiency (7) becomes

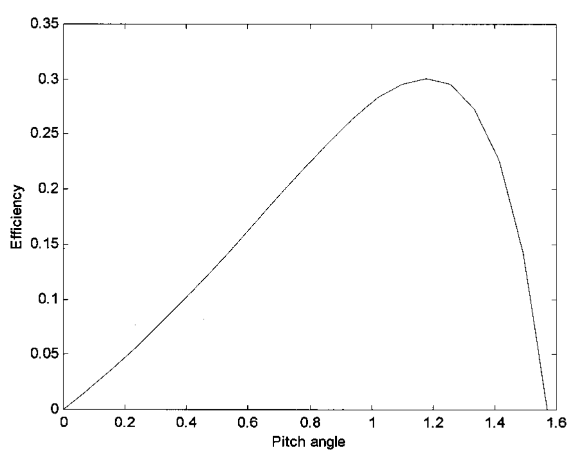


Fig. 6 Efficiency ε versus pitch angle φ

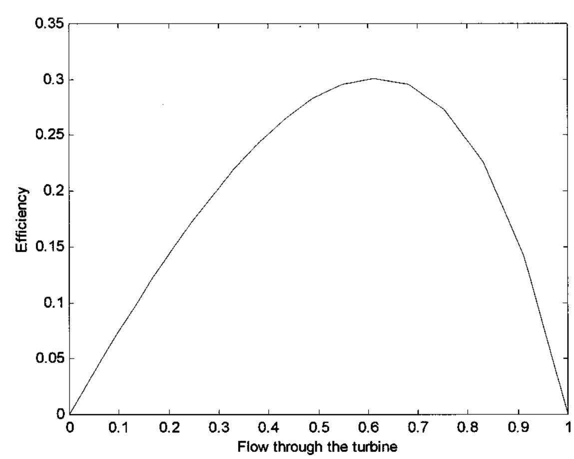


Fig. 8 Efficiency ε versus flow through s

$$\mathcal{E}=\frac{\int_{-1}^{1}V_x(y)(V_\infty^2-V^2)dy}{2V_\infty^3} \tag{17}$$

$$=\frac{\int_{0}^{1}V_x(y)(V_\infty^2-V^2)dy}{V_\infty^3} \tag{18}$$

$$=\int_0^1\left(\mathrm{Re}\frac{dw}{dz}\right)\left(1-\left|\frac{dw}{dz}\right|^2\right)dy \tag{19}$$

$$=\frac{1}{i}\int_0^1\left(\mathrm{Re}\frac{dw}{dz}\right)\left(1-\left|\frac{dw}{dz}\right|^2\right)\frac{dz}{dt}dt \tag{20}$$

$$=s-\frac{1}{i}\int_0^1\left(\mathrm{Re}\frac{dw}{dz}\right)\left|\frac{dw}{dz}\right|^2\frac{dz}{dt}dt \tag{21}$$

$$=s-\sin\varphi\int_0^1\left|\frac{dw}{dz}\right|^3\frac{dz}{i\,dt}dt \tag{22}$$

$$=\frac{1}{I_2(\varphi)}\left(\frac{\varphi}{2}-I_3(\varphi)\sin\varphi\right) \tag{23}$$

where

$$I_3(\varphi)=\frac{I_2(\varphi)}{i}\int_0^1\left|\frac{dw}{dz}\right|^3\frac{dz}{dt}dt \tag{24}$$

$$=\int_0^1(1+\sqrt{1-t^2})^{4\varphi/\pi-2}\times(1-t^2)^{\varphi/\pi}t^{3-6\varphi/\pi}dt \tag{25}$$

6 Computations

The values of the efficiency $\mathcal{E}$ and the fraction s of the flow passing through the turbines are computed numerically for a set of values of φ that are presented in Table 1; their graphs are shown in Figs. 6–8. The pitch angle φ ranges from 0 (the classic Kirchoff flow with complete streamlining) to $\pi/2$ (undisturbed flow). The maximal efficiency is equal to 0.30113 and is attained when $\varphi=3\pi/8=1.17810$ and $s=0.61302$.

7 Conclusions

1 Despite only a narrow class (1-parameter family) of the flows has been considered for optimization, the result obtained allows us to conjecture that the efficiency is maximal when the resistance is rather small and a large part of the flow (61 percent) goes through. In other words, the maximum efficiency could not be noticeably greater than what was obtained here.

2 The model of a free-flow turbine reveals a new class of problems about streamlining with partial penetrating through an obstacle; some of these problems could admit explicit solutions and could have other applications.

3 The velocity of a flow vanishes at the origin of the proposed plane model. This makes the model specifically applicable for two-dimensional propeller-type turbines in free (nonducted) currents. The theoretical limit of the efficiency given by the model is 30.1 percent. A number of tests, as well as constructed power farms, support this thesis in regard to both hydraulic and wind applications. The efficiency of most water and wind propellers in free flows usually ranges from 10 to 20 percent. On the other hand, the three-dimensional hydraulic helical turbine develops an efficiency of about 35 percent in similar free flow conditions [2]. This high efficiency might be explained by modeling a 3-D rotor as a combination of two plane turbines that reflect power contributions from the front and back parts of the original cross-flow turbine.

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