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COMPLEX EIGENVALUE EXTRACTION IN NASTRAN BY THE TRIDIAGONAL REDUCTION (FEER) METHOD

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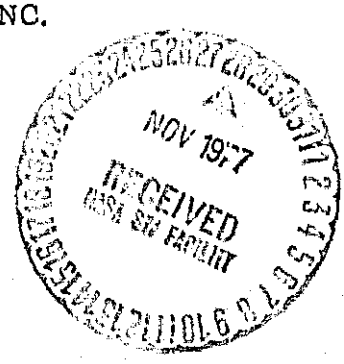

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ABSTRACT

An extension of the Tridiagonal Reduction (FEER) method to complex eigenvalue analysis in NASTRAN is described. As in the case of real eigenvalue analysis, the eigensolutions closest to a selected point in the eigenspectrum are extracted from a reduced, symmetric, tridiagonal eigenmatrix whose order is much lower than that of the full-size problem. The reduction process is effected automatically, and thus avoids the arbitrary lumping of masses and other physical quantities at selected grid points. The statement of the algebraic eigenvalue problem admits mass, damping and stiffness matrices which are unrestricted in character, i.e., they may be real, complex, symmetric or unsymmetric, singular or non-singular.

FOREWORD

The Tridiagonal Reduction method for real eigenvalue analysis (Reference [1]) is extended in the present report to accommodate complex eigenvalue problems, as typified by complex, unsymmetric matrices. The implementation of this technique in NASTRAN permits the extraction of specified numbers of eigenvalues lying closest to selected points in the complex plane, as well as the associated eigenvectors. The eigenvalues may include multiplicities.

Detailed descriptions of the theoretical development, computational procedures, and NASTRAN user instructions are provided.

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SYMBOLS

Latin

$a_{i,j}$, $\bar{\bar{a}}_{i,j}$	general elements of reduced eigenmatrix
d	off-diagonal element of reduced tridiagonal matrix
$\bar{d}$, $\bar{\bar{d}}$	normalization factors for right and left trial vectors and; they are equal and theoretically the same as d
f	number of previously calculated modes
m	size of reduced eigenproblem
n	size of full, unreduced eigenproblem
p, $\bar{p}$	exact and computed physical eigenvalues
$\bar{q}$	desired number of eigensolutions
r	rank of the eigenmatrix, [A]
s	iteration number for successful reorthogonalization
t	number of decimal digits carried in the computations

Greek

Δ	shift value increment for removing dynamic matrix singularities
Λ, $\bar{\Lambda}$	exact and computed values of inverted and shifted eigenvalues
λ_o, λ_o^2	shifts in physical eigenvalues
λ, λ^2	shifted physical eigenvalue
ξ_i	estimate of absolute relative error in i^{th} computed eigenvalue

Matrices and Vectors

$[A]$	eigenmatrix of the full-size problem
$[B]$	damping or pseudo-damping matrix
$[\bar{D}]$	dynamic matrix, $[K+\lambda_o B+\lambda_o^2 M]$
$[H]$	tridiagonal, reduced eigenmatrix
$[I]$	identity matrix
$[K]$	stiffness or pseudo-stiffness matrix
$[L]$	unit lower triangular factor of $[\bar{D}]$
$[M]$	mass or pseudo-mass matrix
$[U]$	upper triangular factor of $[\bar{D}]$
$\{u\}$	modal displacement vector
$[V]$, $[\bar{\bar{V}}]$	matrices consisting of right and left trial vectors
$\{v\}$	modal velocity vector
$\{v_i\}$, $\{\bar{\bar{v}}_i\}$	normalized right and left trial vectors
$\{w_i\}$, $\{\bar{\bar{w}}_i\}$	unnormalized right and left trial vectors
$\{w_r\}$, $\{\bar{\bar{w}}_r\}$	pseudo-random starting or restart vectors
$\{x\}$, $\{\bar{\bar{x}}\}$	exact right and left eigenvectors
$\{\hat{x}\}$, $\{\hat{\bar{\bar{x}}}\}$	calculated right and left eigenvectors
$\{y\}$	eigenvectors of the reduced tridiagonal matrix and participation factors for calculating the physical eigenvectors

1. INTRODUCTION

The complex Tridiagonal Reduction method is an extension of the FEER algorithm (Fast Eigenvalue Extraction Routine) for real eigenvalue analysis to complex, algebraic eigenproblem formulations. A specified number of eigenvalues lying closest to a selected point in the complex plane are sought, as well as the associated eigenvectors. As in the case of real eigenvalue analysis (Reference [1]), these eigensolutions are extracted from a symmetric, tridiagonal eigenmatrix whose order is much lower than that of the full-size problem. In fact, the size of this canonical, reduced matrix is of the same order of magnitude as the number of desired roots, even if the discretized system model possesses thousands of degrees of freedom. The reduction process is carried out via an automatic algorithm requiring a finite number of steps. Thus, a basic weakness of methods requiring the lumping of masses and other physical quantities at arbitrarily selected degrees of freedom (References [2] - [4]) is avoided in reducing the problem size.

With regard to computational speed, the complex Tridiagonal Reduction method is somewhat slower than the Hessenberg method (References [5] and [6]) for small problems (on the order of one hundred or less degrees of freedom), if all the existing eigensolutions are to be calculated. However, it becomes more efficient than the Hessenberg method when the number of requested eigensolutions is much less than the full problem size. Moreover, for much larger problems, the central memory requirement of the Hessenberg method exceeds the capabilities of most large computers, so that it becomes unavailable as a solution option. This limitation does not exist in the case of the Tridiagonal Reduction Method.

As shown in Section 2.1, the complex Tridiagonal Reduction method employs a single initial shift point, and hence only one matrix decomposition is required for each neighborhood chosen in the complex plane. It therefore is more efficient than the complex Inverse Power method, which typically performs many shifts and decompositions for each region selected.

The theory and computational procedures for complex analysis depart from those of real analysis in the following major respects:

1. Both left and right bi-orthogonal vectors must be created in the process of constructing the reduced tridiagonal matrix.

2. The reduced tridiagonal matrix, while symmetric in form, is, in general, complex rather than real.

3. The calculated theoretical errors in the computed eigenvalues are estimates rather than upper bounds.

4. Eigensolutions closest to one or more specified points (shift points) in the complex plane are found. All eigensolutions obtained for previous shift points are swept out of the problem to prevent their regeneration when dealing with the current shift point.

The development of the method is carried out in Section 2. A detailed summary of the computational procedures used in NASTRAN, flow diagrams and user instructions are provided in Sections 3 and 4. NASTRAN user messages and optional diagnostic output are described in Section 5.

2. THEORY FOR COMPLEX EIGENVALUE ANALYSIS

2.1 Problem Formulation

The general complex eigenvalue problem is stated in the form

$$[Mp^2 + Bp + K]\{u\} = 0, \tag{1}$$

where $[M]$, $[B]$, and $[K]$ may be real, complex, symmetric or unsymmetric, singular or non-singular. A specified number of eigenvalues, p, lying closest to a specified point, λ_o, (called a shift point) in the complex plane are to be found, as well as the associated eigenvectors $\{u\}$. The eigenvalues may include multiplicities.

A velocity vector, $\{v\}$, is defined as

$$\{v\} = p\{u\}, \tag{2}$$

and a shifted eigenvalue, λ, is defined as

$$\lambda = p - \lambda_o. \tag{3}$$

Substituting Equations (2) and (3) into Equation (1), the result may be written in partitioned form as

$$\begin{bmatrix} K & B+\lambda_o M \\ -\lambda_o I & I \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \lambda \begin{bmatrix} O & -M \\ I & O \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}, \tag{4}$$

or, using the inverse form, results in

$$[A]\{x\} = \Lambda\{x\}, \tag{5}$$

where

$$[A] = \begin{bmatrix} K & B+\lambda_o M \\ -\lambda_o I & I \end{bmatrix}^{-1} \begin{bmatrix} O & -M \\ I & O \end{bmatrix}, \tag{6}$$

$$\{x\} = \begin{Bmatrix} u \\ -- \\ v \end{Bmatrix} , \tag{7}$$

and

$$\Lambda = \frac{1}{p - \lambda_o} . \tag{8}$$

Equation (5) is an eigenvalue problem in standard form, showing that the order of the eigenvalue problem is doubled due to the presence of the [B] matrix. In the special case where [B] is null (e.g., no damping), the problem formulation becomes

$$[Mp^2 + K]\{u\} = 0, \tag{9}$$

and the double-size eigenvalue problem can be avoided by considering the mathematical eigenvalue to be p^2. Thus, let

$$\lambda^2 = p^2 - \lambda_o^2 , \tag{10}$$

and

$$\Lambda = \frac{1}{\lambda^2} . \tag{11}$$

Substituting Equations (10) and (11) into Equation (9) and using the inverse form results in

$$[K + \lambda_o^2 M]^{-1}[-M]\{u\} = \Lambda\{u\} . \tag{12}$$

Comparing the above with Equation (5) shows that the standard form with a null [B] matrix is achieved by setting

$$[A] = [K + \lambda_o^2 M]^{-1}[-M], \tag{13}$$

and

$$\{x\} = \{u\} . \tag{14}$$

Since the eigenmatrix, $[A]$, is, in general, unsymmetric, the eigenvectors, $\{x\}$, are orthogonal to the eigenvectors, $\{\bar{\bar{x}}\}$, of the transpose eigenproblem

$$[A]^T \{\bar{\bar{x}}\} = \Lambda \{\bar{\bar{x}}\} , \tag{15}$$

so that for $\Lambda_i \neq \Lambda_j$,

$$\{\bar{\bar{x}}_j\}^T \{x_i\} = 0; \quad i \neq j. \tag{16}$$

The above relationship is a biorthogonality condition and the associated eigenvectors, $\{x_i\}$ and $\{\bar{\bar{x}}_j\}$, are called right and left eigenvectors, respectively.

2.2 The Reduction Algorithm

A reduction of the order of the eigenvalue problem, Equation (5), is effected through the transformation

$$\underset{nx1}{\{\hat{x}\}} = \underset{nxm}{[V]} \underset{mx1}{\{y\}} , \tag{17a}$$

and

$$\underset{nx1}{\{\hat{\hat{x}}\}} = \underset{nxm}{[\bar{\bar{V}}]} \underset{mx1}{\{\bar{\bar{y}}\}} , \tag{17b}$$

where $\{\hat{x}\}$ and $\{\hat{\hat{x}}\}$ are approximations of $\{x\}$ and $\{\bar{\bar{x}}\}$, respectively, n is the order of the unreduced problem, and $m \leq n$. The above transformation matrices are chosen to be biorthonormal, so that

$$[\bar{\bar{V}}]^T [V] = [I] . \tag{18}$$

From Equations (5), (17), and (18), it is seen that

$$[H]\{y\} = \bar{\Lambda}\{y\}, \tag{19}$$

where

$$[H] = [\bar{\bar{V}}]^T [A] [V], \tag{20}$$
$$m\times m$$

and $\bar{\Lambda}$ is an approximation of the eigenvalue, Λ.

Thus, Equation (19) is an m^{th} order eigenvalue problem, where $m \leq n$. The value of m is established according to the criteria given in Section 2.3.

As in the case of real eigenvalue analysis (Reference [1]), the Lanczos algorithm is used to construct the transformation matrices vector by vector, i.e.,

$$[V] = [\{v_1\}, \{v_2\}, \ldots, \{v_m\}], \tag{21a}$$
$$n\times m$$

$$[\bar{\bar{V}}] = [\{\bar{\bar{v}}_1\}, \{\bar{\bar{v}}_2\}, \ldots, \{\bar{\bar{v}}_m\}], \tag{21b}$$
$$n\times m$$

such that the reduced $m\times m$ matrix, $[H]$, is tridiagonal and its eigenvalues accurately approximate the roots of Equation (5) having the largest magnitude (or, equivalently, the physical roots, p, closest to the specified point of interest, λ_o, in the complex plane).

The form of the algorithm for generating successive vector pairs is, according to the Lanczos technique,

$$d_{i+1}\{v_{i+1}\} = \{w_{i+1}\} = [A]\{v_i\} - a_{i,1}\{v_1\} - a_{i,2}\{v_2\} \ldots - a_{i,i}\{v_i\}, \tag{22a}$$

$$\bar{\bar{d}}_{i+1}\{\bar{\bar{v}}_{i+1}\} = \{\bar{\bar{w}}_{i+1}\} = [A]^T\{\bar{\bar{v}}_i\} - \bar{\bar{a}}_{i,1}\{\bar{\bar{v}}_1\} - \bar{\bar{a}}_{i,2}\{\bar{\bar{v}}_2\} \ldots - \bar{\bar{a}}_{i,i}\{\bar{\bar{v}}_i\}, \tag{22b}$$

where $\{v_1\}$ and $\{\bar{\bar{v}}_1\}$ are starting vectors (see Section 2.4), $1 \leq i \leq m-1$, and all m vectors pairs are biorthonormal. This implies that

$$a_{i,j} = \{\bar{\bar{v}}_j\}^T [A]\{v_i\}, \tag{23a}$$

and

$$\bar{\bar{a}}_{i,j} = \{v_j\}^T[A]^T\{\bar{\bar{v}}_i\} , \tag{23b}$$

while the normalizing factors are given by

$$d_{i+1} = \bar{\bar{d}}_{i+1} = [\{\bar{\bar{w}}_{i+1}\}^T\{w_{i+1}\}]^{1/2}. \tag{24}$$

Using symmetry arguments similar to those employed for real eigenvalue analysis (Reference [1]), it can be shown that

$$a_{i,i} = \bar{\bar{a}}_{i,i} , \tag{25a}$$

$$a_{i,i-1} = d_i = \bar{\bar{a}}_{i,i-1} , \tag{25b}$$

and

$$a_{i,j} = \bar{\bar{a}}_{i,j} = 0; \quad j<i-1. \tag{25c}$$

The transformed, reduced eigenmatrix in Equation (19) is consequently tridiagonal and symmetric, having the form,

$$[H] = [\bar{\bar{V}}]^T[A][V] = \begin{bmatrix} a_{11} & d_2 & & & \\ d_2 & a_{22} & d_3 & & \\ & d_3 & a_{33} & d_4 & \\ & & \ddots & \ddots & d_m \\ & & & d_m & a_{mm} \end{bmatrix} \tag{26}$$

The matrix coefficients are theoretically given by the simplified recurrence formulas

$$
\left.
\begin{aligned}
a_{i,i} &= \{\bar{\bar{v}}_i\}^T [A] \{v_i\} \\
\{w_{i+1}\} &= [A]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\} \\
\{\bar{\bar{w}}_{i+1}\} &= [A]^T\{\bar{\bar{v}}_i\} - a_{i,i}\{\bar{\bar{v}}_i\} - d_i\{\bar{\bar{v}}_{i-1}\} \\
d_{i+1} &= [\{\bar{\bar{w}}_{i+1}\}^T \{w_{i+1}\}]^{1/2}
\end{aligned}
\right\} \quad i = 1, m \qquad (27a)
$$

$$
\left.
\begin{aligned}
\{v_{i+1}\} &= \frac{1}{d_{i+1}} \{w_{i+1}\} \\
\{\bar{\bar{v}}_{i+1}\} &= \frac{1}{d_{i+1}} \{\bar{\bar{w}}_{i+1}\}
\end{aligned}
\right\} \quad i = 1, m-1 \qquad (27b)
$$

where the sequence is initialized by choosing random, biorthonormal starting vectors for $\{v_1\}$, $\{\bar{\bar{v}}_1\}$ and by setting $d_1 = 0$; $\{v_o\} = \{\bar{\bar{v}}_o\} = \{0\}$.

The final off-diagonal term, d_{m+1}, given by Equation (27a) is used in establishing error estimates for the computed eigenvalues (see Section 2.6). In addition, the above algorithm is modified in the computational scheme as follows:

1. Each pair of vectors $\{v_{i+1}\}$, $\{\bar{\bar{v}}_{i+1}\}$, calculated in Equations (27b) is reorthogonalized to all previously computed pairs, as described in Section 2.5, before re-entering Equations (27a).

2. The size, m, of the reduced problem is a function of the number of accurate eigenvalues requested by the user and is limited to the number of finite physical eigenvalues available (see Section 2.3).

The eigenvalues, $\bar{\Lambda}$, and eigenvectors, $\{y\}$, of Equation (19) are extracted using the Q-R iteration algorithm and eigenvector computational scheme described in connection with the Upper Hessenberg method in NASTRAN (Reference [5]). They are then converted to physical form as follows:

$$\left.\begin{aligned} \bar{p}_i &= \frac{1}{\bar{\Lambda}_i} + \lambda_o \\ \left\{\begin{matrix} u \\ --- \\ v \end{matrix}\right\}_i &= [V]\{y_i\} \end{aligned}\right\} \quad [B] \neq [0], \tag{28a}$$

$$\left.\begin{aligned} \bar{p}_i &= \left(\frac{1}{\bar{\Lambda}_i} + \lambda_o^2\right)^{1/2}; \ \mathrm{Im}(\bar{p}_i) > 0 \\ \{u_i\} &= [V]\{y_i\} \end{aligned}\right\} \quad [B] = [0]. \tag{28b}$$

The velocity vector $\{v_i\}$ in Equation (28a) is discarded prior to further processing of the set of eigensolutions by NASTRAN. Also, all solutions which fail the FEER error test are rejected. However, the number of acceptable solutions will, in all probability, equal or exceed the number requested by the user when the reduced problem size is chosen according to the criteria described in the following section.

2.3 Criteria for the Size of the Reduced Eigenvalue Problem

The maximum number of finite eigensolutions, including any existing rigid body modes, is equal to the rank, r, of the eigenmatrix, [A], in Equation (5). Thus, for example, massless degrees of freedom, appearing as zero diagonal terms in the [M] matrix, will result in singularities (rank reduction), which imply infinite physical eigenvalues. These spurious roots are swept out of the problem in the complex FEER process (see Section 2.4) with a consequent reduction in the available eigensolutions.

A further consideration in limiting the maximum problem size is that the user has the option of requesting eigensolutions in the neighborhood of several shift points ($\lambda_{01}, \lambda_{02}, \ldots$) in the complex plane. In the Tridiagonal Reduction method, all eigensolutions, f, obtained for previous shift points are swept out of the problem to prevent their re-generation when dealing with the current shift point (see Section 2.5). This implies

that the maximum possible size, m, of the reduced problem is further limited to

$$m_{max} = r - f . \tag{29}$$

On the basis of numerical experiments, similar to those cited in Reference [1] for real eigenvalue analysis, it has been found that when $m \ll m_{max}$, a first grouping of more than $m/2$ computed eigenvalues closest to the shift point are in accurate agreement with the corresponding number of exact eigenvalues, provided that $7 \leq m \leq m_{max}$. The remaining reduced-system roots are spread across the remaining exact eigenspectrum. To enhance the accuracy of the associated eigenvectors, the minimum problem size is further increased to twelve, again assuming that $m \ll m_{max}$.

Thus, if the user requests a total of $\bar{q}$ eigenvalues closest to a specified point in the complex plane, the order of the reduced problem is initially set to

$$m \begin{cases} = \min [(2\bar{q}+10), (2n-f)] ; \ [B] \neq [0] , & \text{(30a)} \\ = \min [(2\bar{q}+10), (n-f)] \ ; \ [B] = [0] . & \text{(30b)} \end{cases}$$

Although the total number of eigensolutions requested should not exceed m_{max}, there is usually no simple way to discern this upper limit in complex eigenvalue problems. However, the reorthogonalization tests of Section 2.5 are designed to automatically establish this upper limit. If the latter tests fail for some vector pair $\{v_{i+1}\}$, $\{\bar{v}_{i+1}\}$, this is an indication that a null vector has been generated because m_{max} linearly independent vectors have already been obtained. The recurrence algorithm, Equations (27), is then terminated and the order of the eigenproblem is further reduced to $m = i$.

2.4 Choice of the Initial Trial Vector and Restart Vectors

Because of the inverse relationship between the computed eigenvalues, $\bar{\Lambda}$, and the physical eigenvalues, $\bar{p}$, (see Equations (28)), spurious eigenvalues corresponding to $\bar{\Lambda} = 0$ are equivalent to $\bar{p} \to \infty$. Since these eigenvalues and their corresponding eigenvectors are of no interest and may cause numerical instabilities, they are eliminated from

the reduced tridiagonal problem by employing a constrained set of $\{v\}$, $\{\bar{\bar{v}}\}$ vector pairs having zero projection on the set of eigenvectors associated with $\Lambda = 0$. However, any non-null right and left vectors $\{w_1\}$, $\{\bar{\bar{w}}_1\}$, generated from any other non-null vectors $\{w_r\}$, $\{\bar{\bar{w}}_r\}$ through the transformations

$$\{w_1\} = [A]\{w_r\}, \tag{31a}$$

$$\{\bar{\bar{w}}_1\} = [A]^T\{\bar{\bar{w}}_r\}, \tag{31b}$$

will contain no components of the eigenvectors corresponding to $\bar{\Lambda} = 0$. Further, after the above vectors are orthonormalized

$$\{v_1\} = \frac{1}{[\{\bar{\bar{w}}_1\}^T\{w_1\}]^{1/2}}\{w_1\}, \tag{32a}$$

$$\{\bar{\bar{v}}_1\} = \frac{1}{[\{\bar{\bar{w}}_1\}^T\{w_1\}]^{1/2}}\{\bar{\bar{w}}_1\}, \tag{32b}$$

the next vectors generated,

$$\{w_2\} = [A]\{v_1\} - a_{11}\{v_1\}, \tag{33a}$$

$$\{\bar{\bar{w}}_2\} = [A]^T\{\bar{\bar{v}}_1\} - a_{11}\{\bar{\bar{v}}_1\}, \tag{33b}$$

as well as all subsequent trial vectors will be free of $\bar{\Lambda} = 0$ eigenvectors.

Thus, the starting vectors, $\{v_1\}$, $\{\bar{\bar{v}}_1\}$, in the reduction algorithm are generated from Equations (31) and (32), starting with pseudo-random vectors $\{w_r\}$ and $\{\bar{\bar{w}}_r\}$. If these pseudo-random vectors should, by chance, be deficient in some true eigenvector components, then a null vector $\{v_{i+1}\}$ or $\{\bar{\bar{v}}_{i+1}\}$ may be generated at some point in Equations (27). This is indicated by the appearance of an off-diagonal term, d_{i+1}, which is exceedingly small compared to the corresponding diagonal term, $a_{i,i}$, in the reduced

tridiagonal matrix. The test used is that a null vector has been generated when

$$|d_{i+1}| \leq 10^{-t/2} |a_{i,i}|, \tag{34}$$

where t is the number of decimal digits carried in the computations. In this event, new restart vectors are employed for $\{w_{i+1}\}$, $\{\bar{\bar{w}}_{i+1}\}$. These vectors are generated exactly as in the case of the initial trial vectors, but using different pseudo-random number seeds. The recurrence algorithm for generation of subsequent vector pairs $\{v_{i+1}\}$, $\{\bar{\bar{v}}_{i+1}\}$, etc., is then continued in the usual manner until the required number of vector pairs has been generated.

2.5 Sweeping-out of Previously Obtained Eigenvectors and Reorthogonalization of the Trial Vectors

As in the case of real eigenvalue analysis, successive trial vectors generated by the theoretical algorithm, Equations (27), tend to degrade rapidly as the computations proceed in the finite digit calculations, such that later right vectors are far removed from orthogonality to earlier left vectors, and previously computed eigenvectors, $\{x_1\}$, $\{x_2\}$, ..., $\{x_f\}$; $\{\bar{\bar{x}}_1\}$, $\{\bar{\bar{x}}_2\}$, ..., $\{\bar{\bar{x}}_f\}$. Therefore, each new vector pair $\{v_{i+1}\}$, $\{\bar{\bar{v}}_{i+1}\}$ obtained from either the algorithm or the pseudo-random number generator (see previous Section) is denoted as $\{v_{i+1}^{(0)}\}$, $\{\bar{\bar{v}}_{i+1}^{(0)}\}$ and reorthogonalized with respect to all the previously obtained vectors. This is accomplished by iterating,

$$\{v_{i+1}^{(k)}\} = \{v_{i+1}^{(k-1)}\} - \sum_{j=1}^{i} [\{\bar{\bar{v}}_j\}^T \{v_{i+1}^{(k-1)}\}] \{v_j\} - \sum_{j=1}^{f} [\{\bar{\bar{x}}_j\}^T \{v_{i+1}^{(k-1)}\}] \{x_j\}, \tag{35a}$$

$$\{\bar{\bar{v}}_{i+1}^{(k)}\} = \{\bar{\bar{v}}_{i+1}^{(k-1)}\} - \sum_{j=1}^{i} [\{v_j\}^T \{\bar{\bar{v}}_{i+1}^{(k-1)}\}] \{\bar{\bar{v}}_j\} - \sum_{j=1}^{f} [\{x_j\}^T \{\bar{\bar{v}}_{i+1}^{(k-1)}\}] \{\bar{\bar{x}}_j\}, \tag{35b}$$

where $k = 1, 2, \ldots, s$ until an acceptable vector pair $\{v_{i+1}^{(s)}\}$, $\{\bar{\bar{v}}_{i+1}^{(s)}\}$ is found which satisfies the orthogonality criteria

$$\max_{1 \leq j \leq i} |\{\bar{\bar{v}}_j\}^T \{v_{i+1}^{(s)}\}| \leq 10^{2-t}, \tag{36a}$$

$$\max_{1\le j\le i} |\{v_j\}^T \{\bar{v}_{i+1}^{(s)}\}| \le 10^{2-t}, \tag{36b}$$

$$\max_{1\le j\le f} |\{\bar{x}_j\}^T \{v_{i+1}^{(s)}\}| \le 10^{2-t}, \tag{36c}$$

$$\max_{1\le j\le f} |\{x_j\}^T \{\bar{v}_{i+1}^{(s)}\}| \le 10^{2-t}, \tag{36d}$$

where $s \le 10$, and t is the number of digits carried in the computations. If, for some value of i, the above criteria are not satisfied, then it is assumed that a new pair of trial vectors cannot be generated and a reduced eigenvalue problem of order $m = i$ is solved as discussed in Section 2.3.

If the above criteria are met, then the resulting vectors are biorthonormalized and set equal to the new trial vector pair, i.e.,

$$\{v_{i+1}\} = \frac{1}{[\{\bar{v}_{i+1}^{(s)}\}^T \{v_{i+1}^{(s)}\}]^{1/2}} \{v_{i+1}^{(s)}\}, \tag{37a}$$

$$\{\bar{v}_{i+1}\} = \frac{1}{[\{\bar{v}_{i+1}^{(s)}\}^T \{v_{i+1}^{(s)}\}]^{1/2}} \{\bar{v}_{i+1}^{(s)}\}. \tag{37b}$$

These new vectors are used to compute the next off-diagonal term in the reduced tridiagonal matrix from the formula

$$d_{i+1} = \{\bar{v}_i\}^T [A] \{v_{i+1}\}. \tag{38}$$

However, if

$$|d_{i+1}| \le 10^{-t/2} |a_{i,i}|, \tag{39}$$

it is probable that either $\{v_{i+1}\}$ or $\{\bar{v}_{i+1}\}$ is a null vector. In this event, a reduced eigenvalue problem of order $m = i$ is solved, as above.

If the criteria given by Equations (36) and (39) are both met, then the new, normalized vectors are used to continue the reduction algorithm.

2.6 Error Estimates for the Computed Eigenvalues

Following a development similar to that of Reference [1] for real eigenvalue analysis, it can be shown that

$$||\Lambda_i| - |\bar{\Lambda}_i|| \approx |d_{m+1}\, y_{mi}|. \tag{40}$$

The above shows that the absolute value of the difference between the computed and true eigenvalue magnitudes is proportional to the magnitude of d_{m+1} (which is the next off-diagonal term that would be generated had the reduced tridiagonal matrix, [H], been increased from order m to order m+1) and y_{mi}, (which is the last term in the reduced-system eigenvector associated with $\bar{\Lambda}_i$).

Converting Equation (40) to physical eigenvalue form, using Equations (8), (10), and (11), yields,

$$\left| \frac{|\bar{p}_i - \lambda_o|}{|p_i - \lambda_o|} - 1 \right| \approx \frac{|d_{m+1}\, y_{mi}|}{|\bar{\Lambda}_i|} \; ; \; [B] \neq [0] , \tag{41a}$$

$$\left| \frac{|\bar{p}_i^2 - \lambda_o^2|}{|p_i^2 - \lambda_o^2|} - 1 \right| \approx \frac{|d_{m+1}\, y_{mi}|}{|\bar{\Lambda}_i|} \; ; \; [B] = [0] . \tag{41b}$$

The use of the above error bound estimates as criteria for acceptable eigensolutions is described in Section 3.

3. SUMMARY OF COMPUTATIONAL PROCEDURES AND FLOW CHARTS

Flow diagrams illustrating the computational procedures are shown in Figures 1 and 2. The details of each block are summarized below.

(1) Establish Tentative Reduced Problem Size (see Section 2.3)

Calculate a tentative size, m, of the reduced eigenproblem from

$$m \begin{cases} = \min[(2\bar{q}+10), (2n-f)]; & [B] \neq [0] \quad (1a) \\ = \min[(2\bar{q}+10), (n-f)]; & [B] = [0], \quad (1b) \end{cases}$$

where n is the order of the [K], [B], and [M] matrices, $\bar{q}$ is the number of user requested eigensolutions, and f is the total number of previously extracted eigensolutions calculated for earlier shift points (λ_o or λ_o^2).

(2) Construct Factors of Dynamic Matrix

(a) Set

$$[\bar{D}] \begin{cases} = [K+\lambda_o B+\lambda_o^2 M]; & [B] \neq [0] \quad (2a) \\ = [K+\lambda_o^2 M] \quad ; & [B] = [0]. \quad (2b) \end{cases}$$

(b) Decompose the above dynamic matrix:

$$[\bar{D}] = [L][U], \quad (3)$$

where [L] is a unit lower triangular factor and [U] is an upper triangular factor, using complex arithmetic (subroutine CDCØMP in NASTRAN).

Save the triangular factors for later use in premultiplication operations involving the eigenmatrix, [A], and its transpose. If the decomposition fails, the shift value is increased by $\Delta = .02(1+i)$ and a new decomposition is attempted. If this second

decomposition is unsuccessful, then it is assumed that the problem contains unremovable singularities in the neighborhood of the current shift point and the next region, if any, is examined.

(3) Execute Complex Tridiagonal Reduction Algorithm (see flow diagram for this block, Figure 2)

3.1 Initialize the Recurrence Algorithm (see Section 2.2)

Initialize the vector index to $i=0$ and set

$$\{v_o\} = \{\bar{\bar{v}}_o\} = \{0\}. \tag{4}$$

3.2 Generate a Pair of Starting or Restart Vectors and set $d_{i+1} = 0.0$ (see Section 2.4)

a. Construct right and left vectors $\{w_r\}$ and $\{\bar{\bar{w}}_r\}$ using a pseudo-random number generator. The vectors contain $2n$ elements for $[B] \neq [0]$ and n elements for $[B]=[0]$.

b. Sweep-out components corresponding to $\bar{\Lambda} = 0$:

$$\{w_{i+1}\} = [A]\{w_r\}, \tag{5a}$$

$$\{\bar{\bar{w}}_{i+1}\} = [A]^T\{\bar{\bar{w}}_r\}. \tag{5b}$$

c. Normalize the above vectors:

$$\{v_{i+1}^{(0)}\} = \frac{1}{[\{\bar{\bar{w}}_{i+1}\}^T\{w_{i+1}\}]^{1/2}}\{w_{i+1}\}, \tag{6a}$$

$$\{\bar{\bar{v}}_{i+1}^{(0)}\} = \frac{1}{[\{\bar{\bar{w}}_{i+1}\}^T\{w_{i+1}\}]^{1/2}}\{\bar{\bar{w}}_{i+1}\}. \tag{6b}$$

d. Set $d_{i+1} = 0.0$ and proceed to block 3.5.

3.3 Create One Pair of Approximate Trial Vectors and One Diagonal Coefficient (see Section 2.2)

The recurrence algorithm is:

$$
\left.
\begin{aligned}
a_{i,i} &= \{\bar{\bar{v}}_i\}^T [A] \{v_i\}, \\
\{w_{i+1}\} &= [A]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\}, \\
\{\bar{\bar{w}}_{i+1}\} &= [A]^T\{\bar{\bar{v}}_i\} - a_{i,i}\{\bar{\bar{v}}_i\} - d_i\{\bar{\bar{v}}_{i-1}\}, \\
\bar{d}_{i+1} &= [\{\bar{\bar{w}}_{i+1}\}^T\{w_{i+1}\}]^{1/2}, \\
\{v_{i+1}^{(0)}\} &= \frac{1}{\bar{d}_{i+1}} \{w_{i+1}\}, \\
\{\bar{\bar{v}}_{i+1}^{(0)}\} &= \frac{1}{\bar{d}_{i+1}} \{\bar{\bar{w}}_{i+1}\},
\end{aligned}
\right\} \qquad (7)
$$

where the above vectors are approximations to the new right and left trial vectors.

3.4 First Normalization Test (see Section 2.4)

The test is:

$$|\bar{d}_{i+1}| \geq 10^{-t/2} |a_{i,i}| \qquad (8)$$

Pass: Proceed directly to block 3.5.

Fail: Return to block 3.2, generate a new pair of restart vectors for $\{v_{i+1}^{(0)}\}$ and $\{\bar{\bar{v}}_{i+1}^{(0)}\}$, and proceed to block 5.5.

3.5 Iterate to Obtain Orthogonalized Trial Vectors (see Section 2.5)

Designate $\{x_j\}$, $\{\bar{\bar{x}}_j\}$ (j=1, f) as previously calculated and stored eigenvector pairs, i.e., calculated earlier by complex FEER for previous shift points in the complex plane.

Perform the iterations,

$$\{v_{i+1}^{(k)}\} = \{v_{i+1}^{(k-1)}\} - \sum_{j=1}^{i} [\{\bar{\bar{v}}_j\}^T \{v_{i+1}^{(k-1)}\}]\{v_j\}$$

$$- \sum_{j=1}^{f} [\{\bar{\bar{x}}_j\}^T \{v_{i+1}^{(k-1)}\}]\{x_j\}, \tag{9a}$$

$$\{\bar{\bar{v}}_{i+1}^{(k)}\} = \{\bar{\bar{v}}_{i+1}^{(k-1)}\} - \sum_{j=1}^{i} [\{v_j\}^T \{\bar{\bar{v}}_{i+1}^{(k-1)}\}]\{\bar{\bar{v}}_j\}$$

$$- \sum_{j=1}^{f} [\{x_j\}^T \{\bar{\bar{v}}_{i+1}^{(k-1)}\}]\{\bar{\bar{x}}_j\}; \tag{9b}$$

$k = 1, 2\ldots$, until

$$\left.\begin{aligned} \max_{1 \le j \le i} |\{\bar{\bar{v}}_j\}^T \{v_{i+1}^{(k)}\}| &\le 10^{2-t}, \\ \max_{i \le j \le i} |\{v_j\}^T \{\bar{\bar{v}}_{i+1}^{(k)}\}| &\le 10^{2-t}, \\ \max_{i \le j \le f} |\{\bar{\bar{x}}_j\}^T \{v_{i+1}^{(k)}\}| &\le 10^{2-t}, \\ \max_{i \le j \le f} |\{x_j\}^T \{\bar{\bar{v}}_{i+1}^{(k)}\}| &\le 10^{2-t}, \end{aligned}\right\} \tag{10}$$

or $k = 10$.

If the orthogonality criteria, Equations (10), are satisfied, proceed to block 3.6. Otherwise, set the problem size, m, equal to i and proceed to Exit.

3.6 Normalize the Orthogonalized Trial Vectors (see Section 2.5)

Compute

$$\{v_{i+1}\} = \frac{1}{[\{\bar{\bar{v}}_{i+1}^{(k)}\}^T \{v_{i+1}^{(k)}\}]^{1/2}} \{v_{i+1}^{(k)}\}, \tag{11a}$$

$$\{\bar{\bar{v}}_{i+1}\} = \frac{1}{[\{\bar{\bar{v}}_{i+1}^{(k)}\}^T[v_{i+1}^{(k)}]]^{1/2}} \{\bar{\bar{v}}_{i+1}^{(k)}\}. \tag{11b}$$

These are the new orthogonalized and normalized trial vectors.

3.7 Second Normalization Test and Creation of Off-Diagonal Coefficient (see Section 2.5)

(a) Compute the next off-diagonal term of the reduced tridiagonal matrix from

$$d_{i+1} = \{\bar{\bar{v}}_i\}^T[A]\{v_{i+1}\}. \tag{12}$$

(b) Verify whether the following test is met:

$$|d_{i+1}| \geq 10^{-t/2} |a_{i,i}|. \tag{13}$$

If it is, set i = i+1 and return to block 3.3 for continuation of the recurrence algorithm. If the test fails, set m=i to reduce the problem size and proceed to Exit.

(4) Solve Reduced-System Eigenproblem (see Section 2.2)

(a) The coefficients $a_{11}, a_{22}, \ldots, a_{mm}$ and $d_2, d_3, \ldots, d_m$ computed in block 3 form the diagonal and off-diagonal terms of an m^{th} order symmetric, tridiagonal matrix, [H], (the matrix is, however, usually complex rather than real).

(b) The m^{th} order eigenvalue problem

$$[H]\{y\} = \bar{\Lambda}\{y\} \tag{14}$$

is solved for the eigenvalues, $\bar{\Lambda}_i$, and the eigenvectors, $\{y_i\}$, using the Q-R iteration algorithm and eigenvector computational scheme described in connection with the Upper Hessenberg method in NASTRAN (Reference [5]).

(c) The reduced system eigenvectors are normalized so that

$$\{y_i\}^T\{y_i\} = 1; \quad i = 1, m. \tag{15}$$

(5) Compute Estimate of Eigenvalue Errors (see Section 2.6)

(a) Estimates of absolute relative errors in the computed eigenvalues are obtained from

$$\xi_i = \left| \frac{|\bar{p}_i - \lambda_o|}{|p_i - \lambda_o|} - 1 \right| \approx \frac{|\bar{d}_{m+1} y_{mi}|}{|\bar{\Lambda}_i|}; \quad [B] \neq [0], \tag{16a}$$

$$\xi_i = \left| \frac{|\bar{p}_i^2 - \lambda_o^2|}{|p_i^2 - \lambda_o^2|} - 1 \right| \approx \frac{|\bar{d}_{m+1} y_{mi}|}{|\bar{\Lambda}_i|}; \quad [B] = [0], \tag{16b}$$

where $\bar{d}_{m+1}$ is the last off-diagonal term computed in block 3.3 and y_{mi} is the last element in the vector $\{y_i\}$. If the physical eigenvalue, $\bar{p}_i$, corresponds to a zero root (e.g., a rigid body mode), the above computational scheme is invalid and therefore bypassed. A zero root is assumed to occur whenever

$$\frac{|\bar{p}_i|}{RMS} < 10^{-t/3}, \tag{17}$$

where,

$$RMS = \frac{1}{m}\left[|\bar{p}_1^2| + |\bar{p}_2^2| + \ldots\ldots + |\bar{p}_m^2|\right]^{1/2}, \tag{18}$$

and is denoted by setting the error ξ_i, to zero.

(b) The eigenvalues are listed in order of increasing distance from the shift point, λ_o, to determine whether their associated estimated errors, ξ_i, meet an acceptable relative error tolerance set by the user on the EIGC bulk data card (the default value is $0.10/n$, where n is the order of the stiffness matrix). The first eigenvalue not meeting the tolerance test, as well as all subsequent eigenvalues further removed from the center of interest, are considered to lack sufficient accuracy and are therefore discarded.

(c) Acceptable eigenvalues obtained in the above manner are reordered according to the magnitude of the imaginary part, with positive values considered as a group ahead of all negative values.

(6) Compute Physical Eigenvalues and Eigenvectors and Store (see Section 2.2)

. The mathematical eigenvalues, $\bar{\Lambda}_i$, and eigenvectors, $\{y_i\}$, are converted to physical form as follows:

$$\left.\begin{aligned} \bar{p}_i &= \frac{1}{\bar{\Lambda}_i} + \lambda_o \\ \begin{Bmatrix} u \\ v \end{Bmatrix}_i &= [V]\{y_i\}, \end{aligned}\right\} \quad [B] \neq [0] \tag{19a}$$

$$\left.\begin{aligned} \bar{p}_i &= \left(\frac{1}{\bar{\Lambda}_i} + \lambda_o^2\right)^{1/2}; \quad \mathrm{Im}(\bar{p}_i) > 0 \\ \{u_i\} &= [V]\{y_i\} \end{aligned}\right\} \quad [B] = [0] \tag{19b}$$

where

$$[V] = [\{v_1\}, \{v_2\}, \ldots\ldots, \{v_m\}]. \tag{20}$$

The velocity vector $\{v_i\}$ in Equation (19a) is discarded.

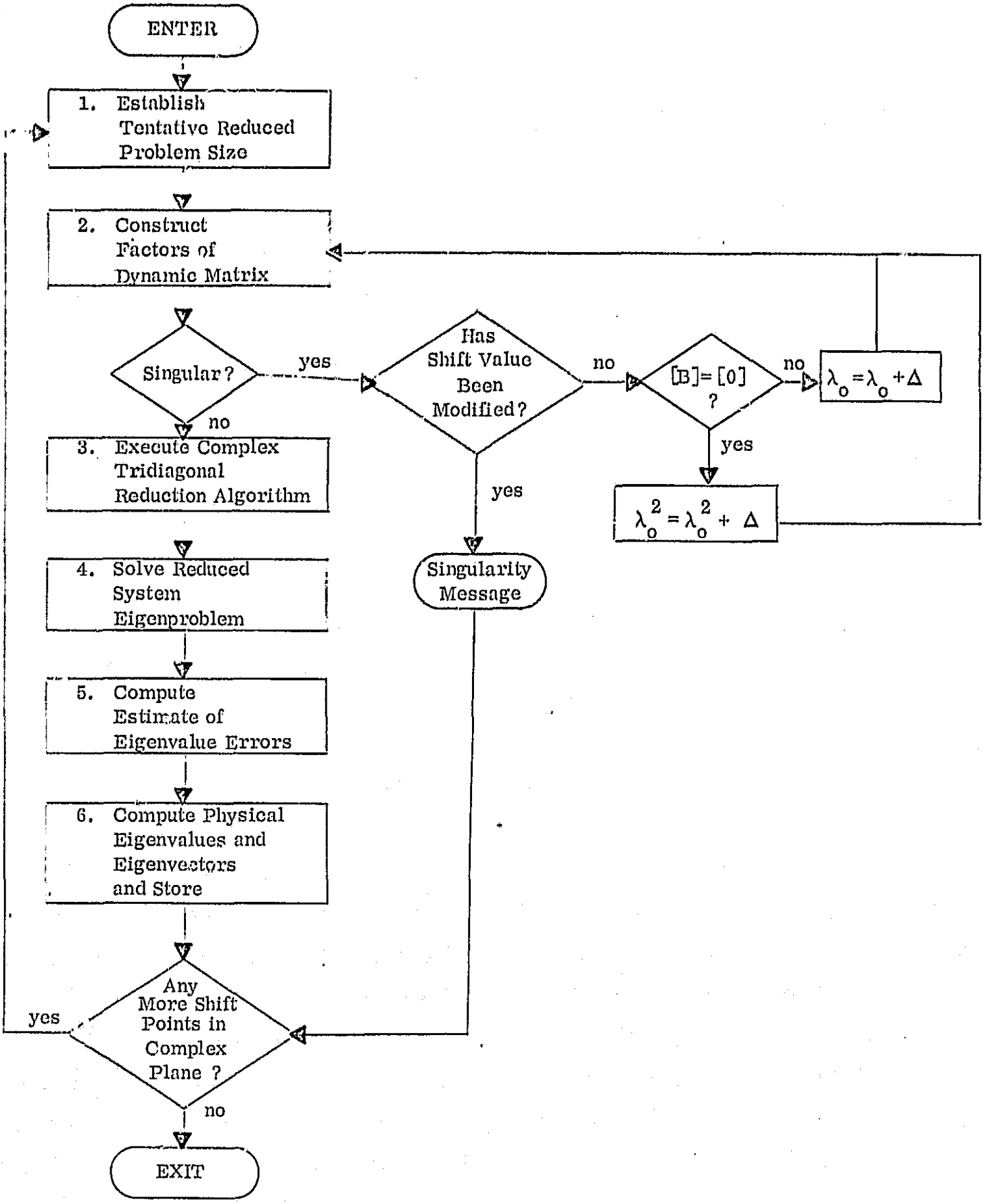


Figure 1. Overall Flow Diagram - Complex Tridiagonal Reduction Method

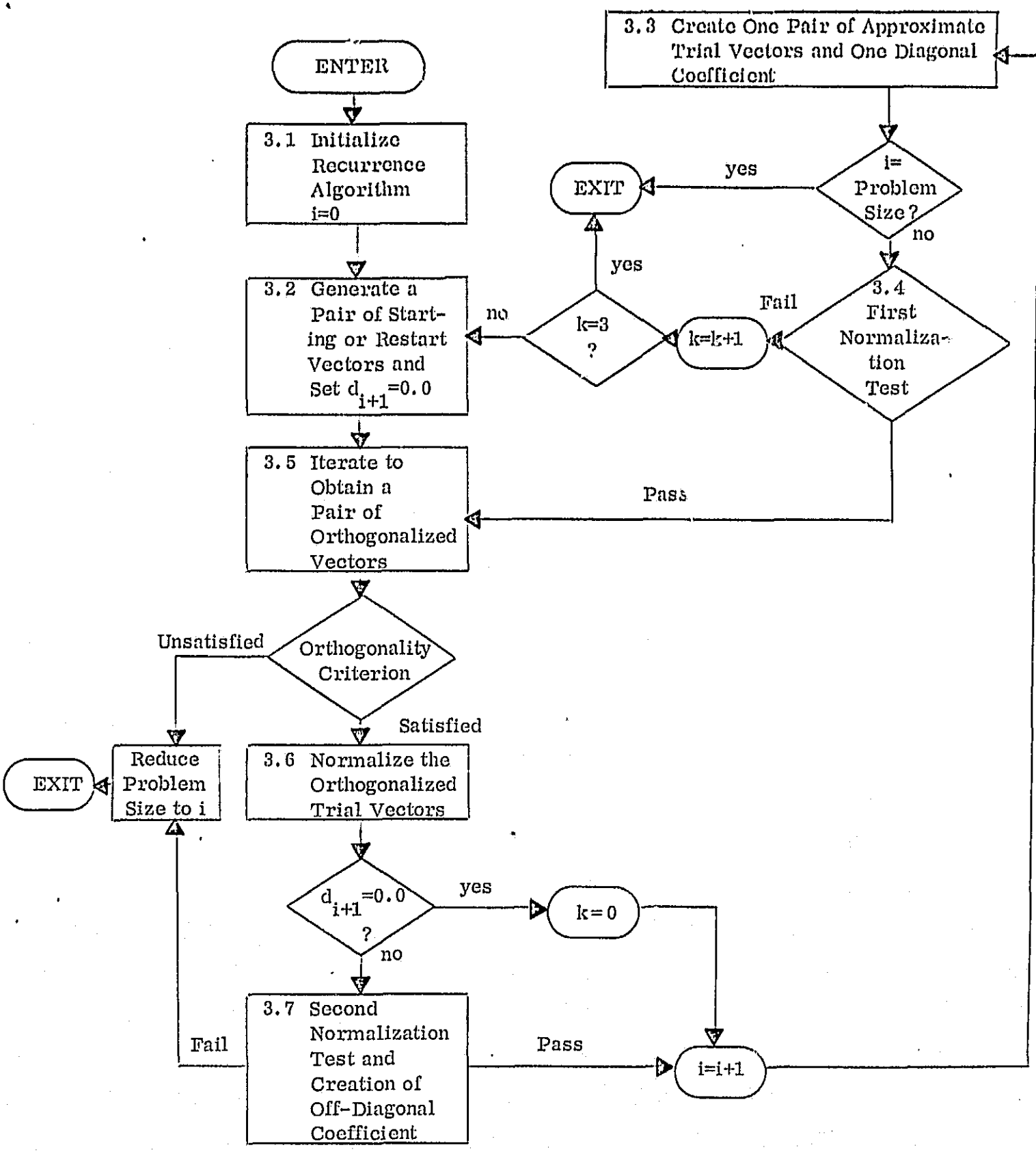


Figure 2. Flow Diagram for Block 3, Execute Complex Tridiagonal Reduction Algorithm.

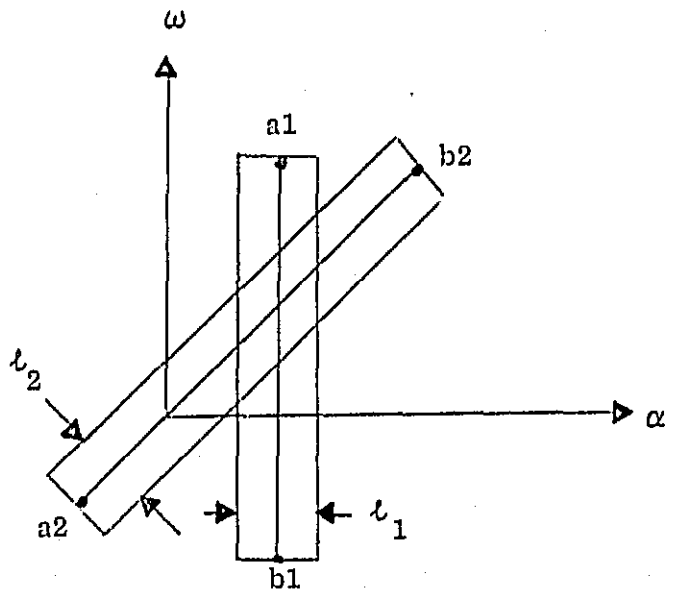
4. NASTRAN USER'S INSTRUCTIONS

The following pages show modifications of the EIGC card in the NASTRAN bulk data deck which accommodate user implementation of the Tridiagonal Reduction method for complex eigenvalue analysis. The modifications consist of additions to the standard user instructions and are underscored for ease in identification.

When the complex Tridiagonal Reduction method is invoked, the E parameter on this card represents the maximum allowable value of the computed absolute relative error in a physical eigenvalue (see Section 2.6). If this value is exceeded, the associated eigensolution is not accepted for further processing by NASTRAN. A detailed list of the maximum relative errors computed by complex FEER can be obtained by requesting DIAG 12 in the NASTRAN Executive Control Deck (see Section 5).

Input Data Card EIGC Complex Eigenvalue Extraction Data

Description: Defines data needed to perform complex eigenvalue analysis



Format and Example:

1	2	3	4	5	6	7	8	9	10
EIGC	SID	METHØD	NØRM	G	C	E			+abc
EIGC	14	DET	PØINT	27		1.-8			ABC
+abc	α_{a1}	ω_{a1}	α_{b1}	ω_{b1}	ℓ_1	N_{el}	N_{d1}		+def
+BC	2.0	5.6	2.0	-3.4	2.0	4	4		DEF
+def	α_{a2}	ω_{a2}	α_{b2}	ω_{b2}	ℓ_2	N_{e2}	N_{d2}		
+EF	-5.5	-5.5	5.6	5.6	1.5	6	3		

(etc.)

Field	Contents
SID	Set identification number (unique integer > 0)
METHØD	Method of complex eigenvalue extraction, one of the BCD values, "INV", "DET", "HESS" or "FEER"
	INV - Inverse power method
	DET - Determinant method
	HESS - Upper Hessenberg method
	FEER - Tridiagonal Reduction Method
NØRM	Method for normalizing eigenvectors, one of the BCD values "MAX" or "PØINT"
	MAX - Normalize to a unit value for the real part and a zero value for the imaginary part, the component having the largest magnitude
	PØINT - Normalize to a unit value for the real part and a zero value for the imaginary part the component defined in fields 5 and 6 - defaults to "MAX" if the magnitude of the defined component is zero.
G	Grid or scalar point identification number (Required if and only if NØRM=PØINT) (Integer > 0)
C	Component number (Required if and only if NØRM="PØINT" and G is a geometric grid point) (0 ≤ integer ≤ 6)
E	Convergence criterion (optional) (Real ≥ 0.0) For method = "FEER", error-tolerance on acceptable eigenvalues in percent (default value is .10/n, where n is the order of the stiffness matrix)
$(\alpha_{aj}, \omega_{aj})$ $(\alpha_{bj}, \omega_{bj})$	Two complex points defining a line in the complex plane (Real) For method = "FEER", $(\alpha_{aj}, \omega_{aj})$ is a point of interest in the complex plane, closest to which the eigenvalues are computed; $\lvert\alpha_{aj}\rvert + \lvert\omega_{aj}\rvert > 0.$ The point $(\alpha_{bj}, \omega_{bj})$ is ignored.
ℓ_j	Width of region in complex plane (Real > 0.0) Blank for method = "FEER".

N_{cj} Estimated number of roots in each region (Integer > 0)
Ignored for method = "FEER".

N_{dj} Desired number of roots in each region (Default is $3N_{cj}$)(Integer > 0)
Desired number of accurate roots for method = "FEER" (Default is 1).

Remarks:

1. Each continuation card defines a rectangular search region. For method = "FEER", the card defines a circular search region, centered at $(\alpha_{aj}, \omega_{aj})$ and of sufficient radius to encompass N_{dj} roots. Any number of regions may be used and they may overlap. Roots in overlapping regions will not be extracted more than once.

2. Complex eigenvalue extraction data sets must be selected in the Case Control Deck (CMETHØD=SID) to be used by NASTRAN.

3. The units of α, ω and ℓ are radians per unit time.

4. At least one continuation card is required.

5. For the determinant method with no damping matrix, complex conjugates of the roots found are not printed.

6. See Section 10.4.4.5 of the Theoretical Manual for a discussion of convergence criteria.

7. For the Upper Hessenberg method, N_{d1} controls the number of vectors computed. Only one continuation card is considered and the (α, ω) pairs, along with the parameters ℓ_1 and N_{c1}, are ignored. Insufficient storage for HESS will cause the program to switch to INV.

8. The error tolerance, E, for the "FEER" method is with regard to

$$\left| \frac{|\bar{p}_i - (\alpha_{aj}, \omega_{aj})|}{|p_i - (\alpha_{aj}, \omega_{aj})|} - 1 \right| \quad \text{for } [B] \neq [0] \quad \text{and}$$

$$\left| \frac{|\bar{p}_i^2 - (\alpha_{aj}, \omega_{aj})^2|}{|p_i^2 - (\alpha_{aj}, \omega_{aj})^2|} - 1 \right| \quad \text{for } [B] = [0],$$

where $\bar{p}_i$ is a computed eigenvalue and p_i an exact eigenvalue.

5. USER MESSAGES AND OPTIONAL DIAGNOSTICS

5.1 NASTRAN Function Module User Messages for the Complex Tridiagonal Reduction Method

5.1.1 General

The following is a description of the NASTRAN user messages which may be generated by NASTRAN during the execution of the Complex Tridiagonal Reduction method and which are unique to this method. Explanatory information is provided following the text of each message and, in the case of a fatal message, corrective action is indicated. Refer to the NASTRAN Users' Manual, Section 6 for a complete listing of other system and user messages.

Fatal messages cause the termination of the execution following the printing of the message text. These messages will always appear at the end of the NASTRAN output. Warning and information messages will appear at various places in the output stream. Such messages convey only warnings or information to the user. Consequently, the execution continues in a normal manner following the printing of the message text.

5.1.2 List of User Messages

3149 *** USER WARNING MESSAGE 3149, USER SPECIFIED NEIGHBØRHØØD CENTERED AT ØRIGIN NØT ALLØWED, CENTER SHIFTED TØ THE RIGHT .001.

Point of interest in the complex plane (α_{ai}, ω_{ai}), closest to which the eigenvalues will be computed, was input as (0.0, 0.0) on an EIGC bulk data continuation card. Since this is an inadmissible choice, the point automatically used was (.001, 0.0).

3150 *** USER WARNING MESSAGE 3150, DESIRED NUMBER ØF EIGENVALUES **** INVALID. SET = 1.

Number of accurate roots desired N_{d1}, was omitted, input as zero or negative on an EIGC bulk data continuation card. The number automatically used was 1.

3151 *** USER WARNING MESSAGE 3151, DYNAMIC MATRIX IS SINGULAR (ØCCURRENCE ****) IN NEIGHBØRHØØD CENTERED AT **** ****

Point of interest in the complex plane (α_{ai}, ω_{ai}), closest to which the

eigenvalues will be computed, was input too close to an eigenvalue on an EIGC bulk data continuation card. The point is automatically shifted by adding .02 to both the real and imaginary parts. If the dynamic matrix is still singular, the next neighborhood, if any, is searched.

3152 *** USER INFØRMATIØN MESSAGE 3152, SUBRØUTINE ALLMAT ØUTPUT EIGENVALUE **** IS NULL.

When an eigenvalue output from subroutine ALLMAT is exactly zero, the formula for computing the associated theoretical error test fails. The magnitude of the eigenvalue is considered to be 10^{-10} for use in that formula.

3153 *** USER WARNING MESSAGE 3153, ATTEMPT TØ NØRMALIZE NULL VECTØR IN SUBRØUTINE CFEER4. NØ ACTIØN TAKEN.

An eigenvector output from subroutine ALLMAT is a zero-vector.

3154 *** USER WARNING MESSAGE 3154, SIZE ØF REDUCED PRØBLEM DECREMENTED ØNCE (NØW ****) DUE TØ NULL ERRØR ELEMENT.

If subroutine CFEER4 receives a reduced tridiagonal matrix having error element d_{m+1} exactly (0, 0), it is impossible to compute meaningful theoretical error estimates for any of the eigenvalues. The size of the reduced problem is reduced by one, so that d_m becomes the new error element.

3155 *** USER WARNING MESSAGE 3155, REDUCED PRØBLEM HAS VANISHED. NØ RØØTS FØUND.

If decrementing the size of the reduced problem (see message 3154) causes the size to become zero, the program continues to the next neighborhood, if any.

3156 *** USER WARNING MESSAGE 3156, SIZE ØF REDUCED PRØBLEM RESTØRED TØ **** BECAUSE NEXT ERRØR ELEMENT WAS ALSØ NULL. ERRØR ELEMENT SET = **** ****

This message follows message 3154. If d_m is also exactly zero (in addition to d_{m+1} being exactly zero), then the original reduced problem size is restored and d_{m+1} is set to ($\mathcal{E}$, 0) where $\mathcal{E}$ = E/100 and E is the error tolerance on acceptable eigenvalues input on the EIGC bulk data card.

3157 *** USER WARNING MESSAGE 3157, FEER PRØCESS MAY HAVE CALCULATED FEWER ACCURATE MØDES **** THAN REQUESTED IN THE NEIGHBØRHØØD ØF **** ****

The desired number of eigenvalues specified on the EIGC bulk data continuation card exceeds the additional number that can be calculated by the Complex Tridiagonal Reduction (Complex FEER) method in the current neighborhood.

3158 *** USER WARNING MESSAGE 3158, NØ ADDITIØNAL MØDES CAN BE FØUND BY FEER IN THE NEIGHBØRHØØD ØF **** ****

An initial pseudo-random vector cannot be made orthogonal to the existing set of orthogonal vectors (which come from Restart and from all prior-neighborhood sets of eigensolutions).

3159 *** USER INFØRMATIØN MESSAGE 3159, ALL SØLUTIØNS HAVE BEEN FØUND.

The FEER method has solved the entire problem. Any additional neighborhoods (as specified by the presence of EIGC bulk data continuation cards) are ignored.

3160 *** USER INFØRMATIØN MESSAGE 3160, MINIMUM ØPEN CØRE NØT USED BY FEER **** WØRDS (**** K BYTES).

This message indicates the amount of open core, in both bytes and words, not used by FEER.

3161 *** USER WARNING MESSAGE 3161, DESIRED NUMBER ØF EIGENSØLUTIØNS **** FØR NEIGHBØRHØØD **** ØF **** CENTERED AT **** **** EXCEEDS THE EXISTING NUMBER ****, ALL EIGENSØLUTIØNS WILL BE SØUGHT.

The desired number of eigenvalues specified on the EIGC bulk data continuation card exceeds the size of the eigenmatrix, which is the maximum possible number of existing eigenvalues.

3162 *** USER WARNING MESSAGE 3162, ATTEMPT TØ NØRMALIZE NULL VECTØR. NØ ACTIØN TAKEN.

The general vector normalization routine (CFNØR1 or CFNØR2) has a zero-vector input to it.

3163 *** USER WARNING MESSAGE 3163, ALL **** SØLUTIØNS HAVE FAILED ACCURACY TEST. NØ RØØTS FØUND.

The number of eigensolutions passing the relative error test is zero. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error bounds could have been obtained by requesting DIAG 12 in the Executive Control Deck.

3164 *** USER INFØRMATIØN MESSAGE 3164, ALL **** SØLUTIONS ARE ACCEPTABLE.

All the eigensolutions obtained in the reduced problem corresponding to the point of interest pass the relative error test. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error estimates could have been obtained by requesting DIAG 12 in the Executive Control Deck.

3165 *** USER INFØRMATIØN MESSAGE 3165, **** SØLUTIØNS HAVE BEEN ACCEPTED AND **** SØLUTIØNS HAVE BEEN REJECTED.

Some eigensolutions passed the relative error test and some did not.

3166 *** USER INFØRMATIØN MESSAGE 3166, **** MØRE ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FØUND FØR NEIGHBØRHØØD **** ØF **** CENTERED AT **** ****. USE DIAG 12 TØ DETERMINE ERRØR ESTIMATES.

The number of eigensolutions passing the relative error test is greater than the number requested on the corresponding EIGC bulk data continuation card. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error estimates could have been obtained by requesting DIAG 12 in the Executive Control Deck.

5.2 The Eigenvalue Summary Table

The following summary of the eigenvalue analysis performed, using the complex Tridiagonal Reduction (FEER) method, is automatically printed:

1. Number of eigenvalues extracted.

2. Number of starting points used.

   This corresponds to the total number of random starting and restart vectors used by the complex FEER process for all neighborhoods.

3. Number of starting point moves.

   Not used in FEER (set equal to zero).

4. Number of triangular decompositions.

   Always equal to the number of points of interest (neighborhoods) in the complex plane processed by FEER, since ordinarily only one triangular decomposition is required by FEER for each point of interest, unless the dynamic matrix is singular at a given point of interest, in which case an additional decomposition is required (obtained by moving the point of interest slightly).

5. Total number of vector iterations.

   The total number of reorthogonalizations of all the trial vectors employed.

6. Reason for termination.

   (0) All, or more solutions than the number requested by the user, have been determined (normal termination).

   (1) All neighborhoods have been processed, but FEER has not obtained the desired number of roots in each neighborhood, possibly because they have already been found in other neighborhoods.

   (2) Abnormal termination - either no roots found or none pass the FEER error test.

5.3 Optional Diagnostic Output

The user can obtain special detailed information relating to the generation of the reduced problem size, the elements of the reduced tridiagonal matrix, vector reorthogonalization iterations, computed error estimates, order of eigenvalue extraction, and distance of extracted eigenvalue from the center of interest by requesting DIAG 12 in the NASTRAN executive control deck.

The meaning of this information is explained below in the order in which it appears in the DIAG 12 output.

****FEER**** (FAST EIGENVALUE EXTRACTIØN RØUTINE)****

This header is always printed first.

****SINGLE PRECISIØN WØRDS ØF ØPEN CØRE NØT USED (SUBRØUTINE XXXX)

**** - Open core not used by subroutine XXXX, in single-precision words.

XXXX - Either CFCNTL, CFEER3, or CFEER4. This message appears three times.

CFCNTL ACCURACY CRITERIØN * (INPUT VALUE **)

* - Accuracy criterion, used for rejecting eigensolutions (expressed as a percentage).

** - Value of accuracy criterion input by the user on the EIGC bulk data card.

CFCNTL NEIGHBØRHØØD * CENTER = ** ** NØ.DES.RTS.= *** NØNSYM = ****

* - Positive integer indicating which neighborhood, or center of interest, is currently being processed.

** ** - Center of interest in the complex plane.

*** - Number of desired roots for the current neighborhood, input by the user on the corresponding EIGC bulk data continuation card.

**** - Indicator which, when nonzero, forces the program to consider the matrices as non-symmetric, even though they may actually be symmetric. This is input by the user in field 7 of each EIGC bulk data continuation card. This input was used during program checkout of the complex FEER process, and it should have no affect on the solution. However, the user should leave field 7 blank on each EIGC continuation card.

REØRTHØGØNALIZATIØN ITERATIØN * TARGET VALUE = **
ERRØRS = *** *** *** ***

* - The reorthogonalization iteration-number. This message will appear many times, as the FEER process "cleans up" each trial vector by forcing it to be as orthogonal as possible to the set of vectors already computed.

** - Convergence tolerance, such that the errors must be smaller than this value. In order to avoid taking square roots, the tolerance and errors are all squared.

*** *** *** *** - Four reorthogonalization errors, the first two of which correspond to the orthogonality of the current right and left handed trial vectors, respectively, with respect to all previously computed vectors in the current neighborhood, and the latter two of which correspond to the orthogonality of the same vectors with respect to all eigenvectors previously computed (restart and prior neighborhoods).

REDUCED TRIDIAGØNAL MATRIX ELEMENTS RØW *
ØFF DIAGØNAL = ** **
DIAGØNAL = *** ***

* - The row number of the (reduced) tridiagonal matrix.

** ** - Value of the off-diagonal element for that row.

*** *** - Value of the diagonal element for that row.

Following the printing of several lines containing reorthogonalization information and reduced tridiagonal matrix elements, when the FEER process has finished its computations for the current point of interest, the header (see above) is printed once again, followed by a table which summarizes all the eigensolutions found by FEER. This table has seven columns, as follows:

(1) Solution number. This is simply a positive integer 1, 2, 3, ...

(2) Order of extraction. These numbers indicate the order in which the tridiagonal matrix was constructed.

(3) Distance from center. This is the distance from the extracted eigenvalue to the neighborhood center (which is printed above the table) in the complex plane. The tabular values are sorted according to increasing distance from the center.

(4) Real part of the extracted eigenvalue.

(5) Imaginary part of the extracted eigenvalue.

(6) Theoretical error estimate. This value must be smaller than the Accuracy Criterion (see above) for the eigensolution to be acceptable.

(7) Status. A single word, "accept" or "reject", to indicate the result of the accuracy test. A minus sign (-) is added to "reject" so that the eye can more rapidly distinguish between the two words.

Finally, this table is printed a second time, but with the rejected eigensolutions deleted.

For very small problems, there is a Very Detailed Printout (VDP) option. This option was originally used to debug the complex FEER logic, and is no longer required. DIAG 12 must be specified in the executive control deck to invoke the VDP option, and furthermore, field 6 of a given EIGC bulk data continuation card must have a (floating point) value equal to or greater than the size of the stiffness matrix. Thus, field 6 of each EIGC continuation card should ordinarily be left blank, so that the VDP option will be suppressed. The actual printed output of this option consists of all vectors for each step of the complex FEER process, which is too extensive and detailed for normal user purposes.

6. REFERENCES

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[4] Ramsden, R.N., and R.J. Stoker, "Mass Condensation: A Semi-automatic Method for Reducing the Size of Vibration Problems", Int. J. Numer. Meth. Eng., Vol. 1, 1969, pp. 333-349.

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